International Journal of Theoretical Physics, Vol. 40, No. 7, 2001

Generalized Uncertainty Relation of One-Dimensional Rindler Oscillator

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Received November 23, 2000

General Minkowski vacuum state is seen to be equivalent to a thermal bath for a Rindler uniformly accelerated observer. This paper calculates the generalized uncertainty relation of one-dimensional Rindler oscillator in the coordinate representation. The calculations show that for a Rindler uniformly accelerated observer there is not only general quantum fluctuation but also thermal fluctuation related to his acceleration.

1. INTRODUCTION

As is well known, an observer at rest in Rindler space is equivalent to a uniformly accelerating observer in Minkowski space, and, in particular, a pure vacuum state of Minkowski representation is a mixed state of finite temperature of Rindler representation (Richard, 1985; Unruh, 1976). In Minkowski vacuum, for a Minkowski observer, coordinates and momentum satisfy general uncertainty relation, whereas for a Rindler observer, for whom Minkowski vacuum may look as a thermal bath (Zhao zheng, 1999), we need to find out what uncertainty relation do its position and momentum satisfy?

With the use of thermal field dynamics (TFD), Mann and his colleagues (Mann *et al.*, 1989; Umezawa and Yamanaka, 1988) present generalized uncertainty relation at first, which is the relation between quantum and thermal fluctuation, and they also give the formal expression of generalized uncertainty relation. Besides, the coordinate representation is usual and important in quantum mechanics. We find that this representation also can be exploited in calculations about thermal nonclassical states, such as coherent state. Minkowski vacuum state can be looked as a coherent state of Bogoliubov-Bardeen-Cooper-Schrieffer type (Shin, 1986) for

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a Rindler observer. In this paper, TFD is introduced into Rindler theory, and generalized uncertainty relation of one-dimensional Rindler oscillator in the coordinate representation is discussed.

2. RINDLER AND MINKOWSKI SPACE–TIME

As we know, the coordinates in Rindler space–time can be obtained from the coordinates in Minkowski space–time under the following coordinates transformation

$$\begin{aligned} T &= a^{-1}e^{a\xi}\sinh a\eta \\ X &= a^{-1}e^{a\xi}\cosh a\eta \quad \text{for region R} \end{aligned} \tag{1}$$

and

$$\begin{aligned} T &= -a^{-1}e^{a\tilde{\xi}}\sinh a\tilde{\eta} \\ X &= -a^{-1}e^{a\tilde{\xi}}\cosh a\tilde{\eta} \quad \text{for region L} \end{aligned} \tag{2}$$

Rindler coordinates (η, ξ) and $(\tilde{\eta}, \tilde{\xi})$ cover space–time region R and L. Region R or L is a quadrant of Minkowski space–time, respectively, as shown in Fig. 1. The region L is called the mirror space–time region of R. With the method of standard Rindler quantization (Birrell and Davies, 1982), we can obtain two groups of annihilation and creation operators $(b, b^{\dagger})$ and $(\tilde{b}, \tilde{b}^{\dagger})$ corresponding to the Rindler modes in the regions R and L, respectively. The vacuum state defined by these two groups of annihilation and creation operators is $|0\rangle_{\text{R}}$ in region R

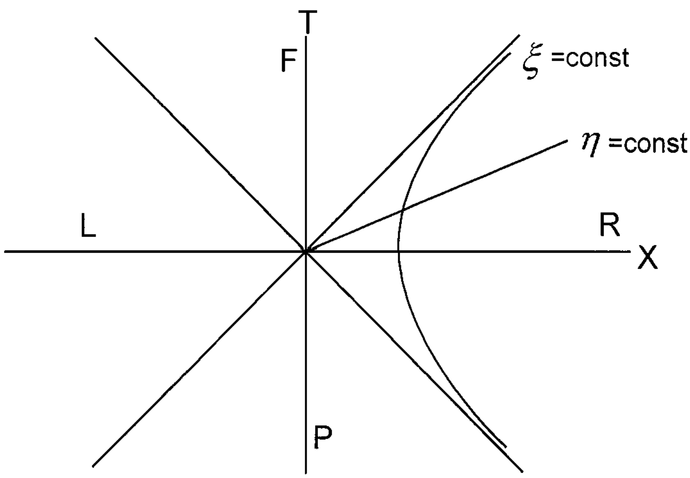

Fig. 1. Rindler coordinatization of Minkowski space.

and $|\tilde{0}\rangle_{\mathrm{R}}$ in region L, respectively. The modes that are corresponding to these two groups of annihilation and creation operators are complete in the regions R and L, respectively, but they are not complete in the whole Mikowski space–time region. In region R, position is denoted by ξ and momentum by p_{R}, while in region L, position is denoted by $\tilde{\xi}$ and momentum by $\tilde{p}_{\mathrm{R}}$.

The Minkowski vacuum is defined by general annihilation and creation operators $(a, a^{\dagger})$. Position is X and momentum is P in Minkowski space–time. For the difference of modes selected, we have two other groups of annihilation and creation operators $(d, d^{\dagger})$ and $(\tilde{d}, \tilde{d}^{\dagger})$. The relation of d, $\tilde{d}$ and Rindler annihilation operators satisfy the Bogoliubov transformation

$$\begin{aligned} d &\equiv T(\theta)\, bT^{\dagger}(\theta) \\ \tilde{d} &\equiv T(\theta)\, \tilde{b}T^{\dagger}(\theta) \end{aligned} \tag{3}$$

where $[d, d^{\dagger}] = [\tilde{d}, \tilde{d}^{\dagger}] = 1$. The unitary transformation (called thermal transformation) is

$$T(\theta) = \exp\{-\theta(\beta)\,(b\tilde{b} - b^{\dagger}\tilde{b}^{\dagger})\} \tag{4}$$

where

$$\tanh\,[\theta(\beta)] = \exp\left(-\frac{\beta\hbar\omega}{2}\right) \tag{5}$$

$\beta = \frac{1}{K_{\mathrm{B}}T}$, with K_{B} the Boltzmann constant and T the temperature. The vacuum state defined by $(d, d^{\dagger})$ and $(\tilde{d}, \tilde{d}^{\dagger})$ is equivalent to the Minkowski vacuum state $|0\rangle_{\mathrm{M}}$, and there are relations as follows

$$d\,|0\rangle_{\mathrm{M}} = \tilde{d}\,|0\rangle_{\mathrm{M}} = 0 \tag{6}$$

For one-dimensional Rindler oscillator, we construct position and momentum $(x,\ p)$ from $(d, d^{\dagger})$ and their tilde conjugate quantities $(\tilde{x}, \tilde{p})$ from $(\tilde{d}, \tilde{d}^{\dagger})$ as follows:

$$\begin{aligned} x &= \sqrt{\frac{\hbar}{2m\omega}}\,(d + d^{\dagger}) \\ p &= -i\sqrt{\frac{m\omega\hbar}{2}}\,(d - d^{\dagger}) \end{aligned} \tag{7}$$

and

$$\begin{aligned} \tilde{x} &= \sqrt{\frac{\hbar}{2m\omega}}\,(\tilde{d} + \tilde{d}^{\dagger}) \\ \tilde{p} &= -i\sqrt{\frac{m\omega\hbar}{2}}\,(\tilde{d} - \tilde{d}^{\dagger}) \end{aligned} \tag{8}$$

The relation between Rindler and Minkowski vacuum is

$$|0\rangle_{\mathrm{M}} = T(\theta)\,|0,\tilde{0}\rangle_{\mathrm{R}} \tag{9}$$

where $|0,\tilde{0}\rangle_{\mathrm{R}}$ is a direct product of the Rindler vacuum state in region R and L, and $T(\theta)$ describes the effect of a thermal bath in which a quantum harmonic oscillator immerse. From Eq. (9), we can say that a thermalizing operator heats the ground state of a zero-temperature harmonic oscillator (Rindler vacuum) into a thermal state with a finite temperature for a Rindler uniformly accelerating observer. Note that any operator in region R commutes with any tilde operator in region L for bosons in this paper. Consequently Minkowski vacuum expectation values for the Rindler observable quantity coincide with its canonical ensemble average in statistical mechanics.

3. GENERALIZED UNCERTAINTY RELATION OF ONE-DIMENSIONAL RINDLER OSCILLATOR

For the quantum one-dimensional oscillator in Rindler space–time region R, its Hamiltonian is

$$H = \frac{1}{2m}p_{\mathrm{R}}^2 + \frac{1}{2}m\omega^2\xi^2 = \left(b^\dagger b + \frac{1}{2}\right)\hbar\omega \tag{10}$$

The wave function of ground state in the coordinate representation is

$$\langle \xi \mid 0\rangle_{\mathrm{R}} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left\{-\frac{m\omega}{2\hbar}\xi^2\right\} \tag{11}$$

where $p_{\mathrm{R}} = -i\hbar\frac{d}{d\xi} \equiv -i\hbar\partial_\xi$, m is the mass, and ω is the angular frequency. And

$$\begin{aligned} b &= \frac{1}{\sqrt{2m\hbar\omega}}(ip_{\mathrm{R}} + m\omega\xi) \\ b^\dagger &= \frac{1}{\sqrt{2m\hbar\omega}}(-ip_{\mathrm{R}} + m\omega\xi) \end{aligned} \tag{12}$$

are the corresponding annihilation and creation operators, respectively. Using the tilde rules in TFD, we introduce the corresponding tilde quantity in space–time region L

$$\tilde{H} = \frac{1}{2m}\tilde{p}_{\mathrm{R}}^2 + \frac{1}{2}m\omega^2\tilde{\xi}^2 = \left(\tilde{b}^\dagger \tilde{b} + \frac{1}{2}\right)\hbar\omega \tag{13}$$

Since the tilde conjugate of a c-number is its complex conjugate and the expectation values of Hermitian operators are real, we obtain

$$\langle\xi\rangle = \langle\tilde{\xi}\rangle, \qquad \langle\xi^2\rangle = \langle\tilde{\xi}^2\rangle,\ \text{etc} \tag{14}$$

According to the invariance of Bogoliubov transformation, we have the identical equation as follows

$$\begin{aligned}\langle(\Delta p)^2\rangle\langle(\Delta x)^2\rangle - \langle\Delta p\Delta\tilde{p}\rangle\langle\Delta x\Delta\tilde{x}\rangle \\ = \langle(\Delta p_{\mathrm{R}})^2\rangle\langle(\Delta\xi)^2\rangle - \langle\Delta p_{\mathrm{R}}\Delta\tilde{p}_{\mathrm{R}}\rangle\langle\Delta\xi\Delta\tilde{\xi}\rangle\end{aligned} \tag{15}$$

where

$$\langle A\rangle = {}_{\mathrm{M}}\langle 0|A|0\rangle_{\mathrm{M}} \tag{16}$$

A is any operator, and $(\Delta A)^2$ is the variance of A. $\langle(\Delta A)^2\rangle = \langle A^2\rangle - \langle A^2\rangle$ is the fluctuation of A. We shall discuss Eq. (15) in the coordinate representation for one-dimensional oscillator.

Substituting Eq. (12) into Eq. (4), one has

$$T(\theta) = \exp\left\{i\frac{\theta}{\hbar}(\xi\tilde{p}_{\mathrm{R}} - \tilde{\xi}p_{\mathrm{R}})\right\} \tag{17}$$

with $\theta \equiv \theta(\beta)$. From Appendix B.4 in Kirzhnits (1967), Eq. (17) can be written as

$$\begin{aligned}T(\theta) = \exp\{-\tanh(\theta)\tilde{\xi}\partial_{\xi}\}\exp\{\ln[\cosh(\theta)](\xi\partial_{\xi} - \tilde{\xi}\partial_{\tilde{\xi}})\} \\ \times\exp\{-\tanh(\theta)\xi\partial_{\tilde{\xi}}\}\end{aligned} \tag{18}$$

Using the following operator properties

$$e^{C\partial_y}f(y) = f(y + C) \tag{19}$$

and

$$e^{C_y\partial_y}f(y) = f(ye^C) \tag{20}$$

one can give the wave function of Minkowski vacuum in Rindler coordinate representation

$$\begin{aligned}\langle\tilde{\xi},\xi \mid 0\rangle_{\mathrm{M}} &= T(\theta)\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left\{-\frac{m\omega}{2\hbar}(\xi^2 + \tilde{\xi}^2)\right\} \\ &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left\{-\frac{m\omega}{2\hbar}\left[\begin{matrix}(\xi\cosh(\theta) - \tilde{\xi}\sinh(\theta))^2 \\ +(\tilde{\xi}\cosh(\theta) - \xi\sinh(\theta))^2\end{matrix}\right]\right\}\end{aligned} \tag{21}$$

When $\beta \to \infty$, from Eqs. (4) and (5) $\theta(\beta) \to 0$, $T(\theta) \to 1$, so $\langle\tilde{\xi},\xi \mid 0\rangle_{\mathrm{M}}$ is reduced to $\langle\tilde{\xi},\xi \mid 0,\tilde{0}\rangle_{\mathrm{R}}$.

By a Bogoliubov transformation, we can get a simple formula of Eq. (21):

$$\langle\tilde{\xi},\xi \mid 0\rangle_{\mathrm{M}} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left\{-\frac{m\omega}{2\hbar}(x^2 + \tilde{x}^2)\right\} \tag{22}$$

where

$$x \equiv T(\theta)\xi T^{\dagger}(\theta)$$
$$\tilde{x} \equiv T(\theta)\tilde{\xi} T^{\dagger}(\theta) \tag{23}$$

Next, we shall calculate the fluctuation of one-dimensional Rindler oscillator in the coordinate representation in the thermal equilibrium. Now first we calculate the Rindler position probability density in Minkowski vacuum.

$$\begin{aligned}
&\rho_{\xi'\tilde{\xi}'\xi\tilde{\xi}} \\
&\quad = {}_{\mathrm{M}}\langle 0 \mid \xi', \tilde{\xi}'\rangle\langle\tilde{\xi}, \xi \mid 0\rangle_{\mathrm{M}} \\
&\quad = \frac{m\omega}{\pi\hbar} \exp\left\{-\frac{m\omega}{2\hbar}\left[\begin{aligned}&\xi'^2(\sinh^2(\theta)+\cosh^2(\theta)) - 4\xi'\tilde{\xi}'\sinh(\theta)\cosh(\theta)\\ &\quad +\tilde{\xi}'^2(\sinh^2(\theta)+\cosh^2(\theta)) + \xi^2(\sinh^2(\theta)+\cosh^2(\theta))\\ &\quad - 4\xi\tilde{\xi}\sinh(\theta)\cosh(\theta) + \tilde{\xi}^2(\sinh^2(\theta)+\cosh^2(\theta))\end{aligned}\right]\right\}
\end{aligned} \tag{24}$$

Taking $\xi = \xi'$ and $\tilde{\xi} = \tilde{\xi}'$ in Eq. (24), we obtain

$$\rho_{\xi\tilde{\xi}\xi\tilde{\xi}} = \frac{m\omega}{\pi\hbar} \exp\left\{-\frac{m\omega}{\hbar}\left[\begin{aligned}&\xi^2(\sinh^2(\theta)+\cosh^2(\theta)) - 4\xi\tilde{\xi}\sinh(\theta)\cosh(\theta)\\ &\quad +\tilde{\xi}^2(\sinh^2(\theta)+\cosh^2(\theta))\end{aligned}\right]\right\} \tag{25}$$

This is a Gaussian probability density, and from it one can easily get

$$\langle\Delta\xi\,\Delta\tilde{\xi}\rangle = \int_{-\infty}^{+\infty} \xi\tilde{\xi}\,\rho_{\xi\tilde{\xi}\xi\tilde{\xi}}\, d\xi\, d\tilde{\xi} = \frac{\hbar}{m\omega}\sinh(\theta)\cosh(\theta) \tag{26}$$

Similar to the position, we can get some corresponding results for momentum

$$\begin{aligned}
\rho_{p_{\mathrm{R}},\tilde{p}_{\mathrm{R}},p_{\mathrm{R}},\tilde{p}_{\mathrm{R}}} &= \int_{-\infty}^{+\infty} \frac{1}{2\pi\hbar} \exp\left\{i\frac{p_{\mathrm{R}}\xi'}{\hbar} - i\frac{p_{\mathrm{R}}\xi}{\hbar} + i\frac{\tilde{p}_{\mathrm{R}}\tilde{\xi}'}{\hbar} - i\frac{\tilde{p}_{\mathrm{R}}\tilde{\xi}}{\hbar}\right\}\\
&\quad \times \rho_{\xi'\tilde{\xi}'\xi\tilde{\xi}}\, d\xi\, d\xi'\, d\tilde{\xi}\, d\tilde{\xi}'\\
&= c\frac{\pi^2}{a^2 - \frac{1}{4}b^2} \exp\left\{-\frac{1}{2\hbar^2}\frac{ap_{\mathrm{R}}^2 + bp_{\mathrm{R}}\tilde{p}_{\mathrm{R}} + a\tilde{p}_{\mathrm{R}}^2}{a^2 - \frac{1}{4}b^2}\right\}
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
a &= \frac{m\omega}{2\hbar}[\sinh^2(\theta) + \cosh^2(\theta)]\\
b &= \frac{2m\omega}{\hbar}\sinh(\theta)\ \cosh(\theta)\\
c &= \frac{m\omega}{4\pi^3\hbar^3}
\end{aligned} \tag{28}$$

Hence

$$\langle \Delta p_{\mathrm{R}} \Delta \tilde{p}_{\mathrm{R}} \rangle = \int_{-\infty}^{+\infty} p_{\mathrm{R}} \tilde{p}_{\mathrm{R}} \rho_{p_{\mathrm{R}}, \tilde{p}_{\mathrm{R}}, p_{\mathrm{R}}, \tilde{p}_{\mathrm{R}}} dp_{\mathrm{R}}\, d\tilde{p}_{\mathrm{R}} = \hbar\omega m \sinh(\theta) \cosh(\theta) \quad (29)$$

so we can obtain the second term on the right hand side of Eq. (15)

$$\langle \Delta\xi \Delta\tilde{\xi} \rangle \langle \Delta p_{\mathrm{R}} \Delta \tilde{p}_{\mathrm{R}} \rangle = \hbar^2 \sinh^2(\theta) \cosh^2(\theta) = \frac{\hbar^2}{4 \sinh^2\left(\frac{\beta\omega\hbar}{2}\right)} \quad (30)$$

Noting

$$\begin{aligned} \langle \Delta p \Delta \tilde{p} \rangle &= \langle \Delta p \rangle \langle \Delta \tilde{p} \rangle = 0 \\ \langle \Delta x \Delta \tilde{x} \rangle &= \langle \Delta x \rangle \langle \Delta \tilde{x} \rangle = 0 \end{aligned} \quad (31)$$

Equation (15) can be turned into

$$\langle (\Delta p_{\mathrm{R}})^2 \rangle \langle (\Delta\xi)^2 \rangle = \langle (\Delta p)^2 \rangle \langle (\Delta x)^2 \rangle + \langle \Delta p_{\mathrm{R}} \Delta \tilde{p}_{\mathrm{R}} \rangle \langle \Delta\xi \Delta\tilde{\xi} \rangle \quad (32)$$

Thus, we have the Generalized Uncertainty Relation of one-dimensional Rindler oscillator in the coordinate representation in Minkowski vacuum

$$\langle (\Delta p_{\mathrm{R}})^2 \rangle \langle (\Delta\xi)^2 \rangle \geq \frac{\hbar^2}{4} + \frac{\hbar^2}{4 \sinh^2\left(\frac{\beta\omega\hbar}{2}\right)} \quad (33)$$

4. SUMMARY AND DISCUSSION

The key result of this paper is Eq. (33), which describes the relation of quantum fluctuation, thermal fluctuation, and total fluctuations of one-dimensional Rindler oscillator in the coordinate representation and separates thermal fluctuation from quantum ones intuitionisticly. For a Rindler uniformly accelerated observer, the term on the left hand side of Eq. (33) describes total fluctuations of one-dimensional Rindler oscillator. The first term on the right hand side of Eq. (33) $\langle (\Delta p)^2 \rangle \langle (\Delta x)^2 \rangle$ describes zero-temperature fluctuation, which is purely a quantum fluctuation and satisfies the general uncertainty relation.

For a Rindler uniformly accelerated observer, the second term on the right hand side of Eq. (33) describes the purely thermal fluctuation of one-dimensional Rindler oscillator, which is determined by cross terms of tilde and nontilde operators. When $T \to 0$, $\beta \to \infty$, the thermal fluctuation $\langle \Delta p_{\mathrm{R}} \Delta \tilde{p}_{\mathrm{R}} \rangle \langle \Delta\xi \Delta\tilde{\xi} \rangle \to 0$. It describes that the thermal fluctuation approaches 0 at zero-temperature. This phenomenon can be theoretically interpreted: from the Rindler effect Minkowski vacuum can be seen as a thermal bath for an uniformly accelerated observer and temperature is proportional to his acceleration. When $T \to 0$, the acceleration of the Rindler uniformly accelerated observer $a \to 0$. In this case, the Rindler uniformly accelerated observer is just the general Minkowski inertial observer. At this time total fluctuations reduce to quantum fluctuation, and are proportional to $\hbar$.

In addition, Rindler radiation is a kind of purely quantum effect. When $\hbar \to 0$, quantum effect is not considered, and Rindler effect will not exist. While discussing the fluctuation of one-dimensional Rindler oscillator, we find that when $\hbar \to 0$, thermal fluctuation of one-dimensional Rindler oscillator still exists, and it is not related to $\hbar$.

$$\langle \Delta p_{\mathrm{R}} \Delta \tilde{p}_{\mathrm{R}} \rangle \langle \Delta \xi \Delta \tilde{\xi} \rangle \to \frac{1}{\beta^2 \omega^2} = \frac{k^2 T^2}{\omega^2} \tag{34}$$

This describes that thermal fluctuation of one-dimensional Rindler oscillator is proportional to T^2 and inverse to ω^2 without considering quantum effect.

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